ON THE THEORY OF DIFFERENTIAL GAMES

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We develop lemmas which clarify the possibility of approximating extremal strategies in a differential game involving encounter with a specified set M. The study constitutes an extension of [1-14].

1. Let us consider a controlled system described by the differential equation

$$dx/dt = f^{(1)}(t, x, u) + f^{(2)}(t, x, v) \tag{1.1}$$

Here x is the n-dimensional phase vector of the system; u, v are the r-dimensional vectors of the controlling forces at the disposal of the first and second player, respectively. The realizations $u[t]$ and $v[t]$ of the controls u and v are restricted by the condition

$$u[t] \in U, \qquad v[t] \in V \tag{1.2}$$

where U and V are bounded closed sets; the functions $f^{(i)}$ are continuous in all their arguments and satisfy the Lipschitz conditions in x. (The argument t will be enclosed in square bracket to indicate that we are dealing with the realizations of the corresponding functions during the playing of the game; as a rule, this argument will be enclosed in parentheses in expressing functions which occur in ancillary constructions).

Our purpose in the present paper is to discuss certain aspects of the conflict problem of the convergence of the point $x[t]$ to a specified closed set M, the first player's aim being to achieve convergence and the second player's aim to avoid it. Let us define the problem in more specific terms.

We shall be concerned with strategies $\mathbf{U}$ forming the controls $u_\Delta[t]$ which remain constant over certain sufficiently small half-intervals $[\tau_i, \tau_{i+1})$. We shall say that we have chosen a strategy (an approximating strategy $\mathbf{U}$) if for every sufficiently small $\Delta > 0$ each pair $\{t, x\}$ $(t_0 \leqslant t < \vartheta,\ -\infty < x_i < \infty,\ i = 1, 2, \ldots, n)$ is associated with some set $U_\Delta(t, x) \subset U$. The strategy $\mathbf{U}$ forms the Δ-controls $u_\Delta[t]$ in the following way.

We introduce the half-interval $[t_0, \vartheta)$ covered by the system of half-intervals $[\tau_i, \tau_{i+1})$ $(i = 0, 1, 2, \ldots;\ \tau_0 = t_0,\ \max(\tau_{i+1} - \tau_i) = \Delta)$. Then

$$u_\Delta[t] = u_\Delta[\tau_i] \in U_\Delta(\tau_i, x[\tau_i]) \qquad (\tau_i \leqslant t < \tau_{i+1}) \tag{1.3}$$

where $x[\tau_i]$ is the phase state of system (1.1) which is realized at the instant $t = \tau_i$ through the action of the control $u_\Delta[t]$ and of some control $v[t]$ realized by the second player over the preceding half-interval $[t_0, \tau_i)$. The choice of the control $u_\Delta[\tau_i]$ from the set $U_\Delta(\tau_i, x[\tau_i])$ remains arbitrary.

We shall use the symbol $\rho(x, M)$ to denote the distance from the point x to the set M. The integrable functions $u(t)$ and $v(t)$ satisfying the conditions $u(t) \in U$ and $v(t) \in V$ will be called "permissible".

We say that for a given initial condition $x[t_0] = x_0$ the strategy $\mathbf{U}$ guarantees convergence of the point $x[t]$ to the set M at an instant ϑ if the motions $x[t]$ generated by this strategy satisfy the condition

$$\lim_{\Delta\to 0} \sup_{v[t]} \sup_{u_\Delta} \rho(x[\vartheta], M) = 0 \tag{1.4}$$

We say that for a given $x[t_0] = x_0$ the strategy $\mathbf{U}$ guarantees the convergence of the point $x[t]$ to the set M not later than at an instant $\vartheta > t_0$ if the motions $x[t]$ generated by this strategy satisfy the condition

$$\lim_{\Delta\to 0} \sup_{v[t]} \sup_{u_\Delta} \inf_{t} \rho(x[t], M) = 0 \tag{1.5}$$

Here the upper bounds must be computed over all the possible permissible realizations $v[t]$ and over all possible controls $u_\Delta[t]$ (1.3) corresponding to a given strategy $\mathbf{U}$; the lower bound in (1.5) must be computed over all the t from the segment $t_0 \leqslant t \leqslant \vartheta$.

The problem consists in constructing a strategy $\mathbf{U}$ which will guarantee convergence of the point $x[t]$ to the set M at some instant ϑ or not later than at some instant ϑ.

2. Let us begin by defining some terms and symbols. Let each value t from some segment $t_0 \leqslant t \leqslant \vartheta$ be associated with a closed set $W(t)$ in the space $\{x\}$. We say that the sets $W(t)$ $(t_0 \leqslant t \leqslant \vartheta)$ are strongly stable if the following condition is fulfilled.

Condition 2.1. For any values of t_* from the half-interval $[t_0, \vartheta)$ any point x_* from the set $W(t_*)$, and any number Δ from the half-interval $(0, \min\{\Delta_0, \vartheta - t_*\}]$ (where Δ_0 is a sufficiently small positive constant) it is possible to choose for any permissible function $v(t)$ $(t_* \leqslant t < t_* + \Delta)$ a permissible function $u(t)$ $(t_* \leqslant t < t_* + \Delta)$ such that the pair of controls $\{u(t), v(t)\}$ carries system (1.1) from the position $x(t_*) = x_*$ to the state $x(t_* + \Delta) \in W(t_* + \Delta)$.

Let $M \subset W(t)$. We call the sets $W(t)$ $(t_0 \leqslant t \leqslant \vartheta)$ "stable" if the following condition is fulfilled.

Condition 2.2. For any values of t_* from the half-interval $[t_0, \vartheta)$, any point x_* from the set $W(t_*)$, and any number Δ from the half-interval $(0, \min\{\Delta_0, \vartheta - t_*, \xi\rho(x_*, M)\}]$ (where Δ_0 and ξ are sufficiently small positive constants) it is possible to choose for any permissible function $v(t)$ $(t_* \leqslant t < t_* + \Delta)$ a permissible function $u(t)$ $(t_* \leqslant t < t_* + \Delta)$ such that the pair of controls $\{u(t), v(t)\}$ carries system (1.1) from the position $x(t_*) = x_*$ to the state $x(t_* + \Delta) \in W(t_* + \Delta)$.

We introduce the following notation:

$$\varepsilon^\circ(\mathbf{U}) = \lim_{\Delta\to 0} \sup_{v(t)} \sup_{u_\Delta} \sup_{t} \rho(x[t], W(t)) \tag{2.1}$$

$$\varepsilon_0(\mathbf{U}) = \lim_{\Delta\to 0} \sup_{v(t)} \sup_{u_\Delta} \inf_{t} \rho(x[t], M) \tag{2.2}$$

Here the lower or upper bounds must be computed over all t from the segment $[t_0, \vartheta]$, over all permissible realizations $v(t)$, and over all possible Δ-controls $u_\Delta[t]$ (1.3) corresponding to the strategy $\mathbf{U}$.

Let us construct the extremal approximating strategy $\mathbf{U}^{(e)}$ based on the systems of sets $W(t)$ satisfying either Condition 2.1 or 2.2. The sets $U_\Delta^{(e)}(t, x)$ corresponding to the strategy $\mathbf{U}^{(e)}$ can be constructed as follows. If the point x lies in the set $W(t)$, we set

$$U_\Delta^{(e)}(t, x) = U \tag{2.3}$$

If the point x does not lie in the set $W(t)$, we proceed as follows. We isolate from the totality of systems $W(t)$ the set Q of all points q lying closest to the point x. The symbol S denotes the set of all unit vectors s directed from the point x to the points q from Q. As our $U_\Delta^{(e)}(t, x)$ we choose the set of all vectors $u = u^e$ from U which satisfy the condition

$$s'f^{(1)}(t, x, u^e) = \max s'f^{(1)}(t, x, u) \qquad (u \in U) \tag{2.4}$$

for at least one s from S (the primes are used here and below to indicate transposition).

Thus, according to (2.4) the control $u_\Delta[t]$ (1.3) is selected by the extremal strategy $\mathrm{U}^{(e)}$ from the condition of maximum possible displacement of the phase point $x[t]$ towards the set $W(t)$.

3. The following statement is valid.

Lemma 3.1. Let $x[t_0] = x_0 \in W(t_0)$. If the sets $W(t)$ $(t_0 \leqslant t \leqslant \vartheta)$ are strongly stable, then the extremal strategy $\mathrm{U}^{(e)}$ ensures that

$$\varepsilon^\circ(\mathrm{U}^{(e)}) = 0 \tag{3.1}$$

In other words, if the sets $W(t)$ are strongly stable for all sufficiently small values of $\Delta > 0$, then the controls $u_\Delta[t]$ chosen by way of the extremal strategy $\mathrm{U}^{(e)}$, ensure retention of the phase point $x[t]$ $(t_0 \leqslant t \leqslant \vartheta)$ in an arbitrarily small prescribed neighborhood of the sets $W(t)$ $(t_0 \leqslant t \leqslant \vartheta)$ regardless of the actions of the second player.

Verification of condition (3.1) is based on the following estimate. Let us write $\varepsilon[t] = \rho(x[t], M)$, where $x[t]$ is the motion generated by the control $u_\Delta^e[t]$ corresponding to the strategy $\mathrm{U}^{(e)}$ and by some arbitrary permissible control realized in the form of an integrable function $v_*[t] \in V$. Let the state $x[\tau_i]$ be realized at some instant $t = \tau_i$, and let the point $x[\tau_i]$ lie outside the set $W(\tau_i)$.

Let us assume for the time being that at the instant $t = \tau_i$ system (1.1) finds itself in the state $x = x_*(\tau_i) = q$, where q is precisely the point from the set Q towards which the vector s of condition (2.4) is pointing (at $t = \tau_i$ and $x = x[\tau_i]$).

Then, by Condition 2.1, the control $v_*[t]$ $(\tau_i \leqslant t < \tau_{i+1})$ is associated with a control $u_*(t)$ $(\tau_i \leqslant t < \tau_{i+1})$ such that the pair of controls $\{u_*(t), v_*[t]\}$ carries system (1.1) from the position $x_*(\tau_i) = q \in W(\tau_i)$ to the state $x_*(\tau_{i+1}) \in W(\tau_{i+1})$.

We infer from this that the same pair of controls $\{u_*(t), v_*[t]\}$ would carry system (1.1) from the actually realized state $x[\tau_i]$ to a state $x^*(\tau_{i+1})$ such that

$$\rho(x^*(\tau_{i+1}), W(\tau_{i+1})) \leqslant \varepsilon[\tau_i] + \lambda\varepsilon[\tau_i](\tau_{i+1} - \tau_i) + o(\tau_{i+1} - \tau_i) \tag{3.2}$$

where λ is a constant and where the symbol $o(\alpha)$ denotes an infinitely small quantity of order higher than that of α. But replacement of the control $u_*(t)$ by the control

$$u_\Delta^{(e)}[t] = u_\Delta^{(e)}[\tau_i] \in U_\Delta^{(e)}(\tau_i, x[\tau_i]) \quad (\tau_i \leqslant t < \tau_{i+1})$$

which satisfies maximum condition (2.4) can result in a deterioration of estimate (3.2) by an amount not exceeding $o(\tau_{i+1} - \tau_i)$. We therefore have the inequality

$$\rho(x[\tau_{i+1}], W(\tau_{i+1})) = \varepsilon[\tau_{i+1}] \leqslant \varepsilon[\tau_i] + \lambda\varepsilon[\tau_i](\tau_{i+1} - \tau_i) + o(\tau_{i+1} - \tau_i) \tag{3.3}$$

which implies the validity of Lemma 3.1. (We must bear in mind that the estimate

$$\rho(x[\tau_i], x[\tau_{i+1}]) = O(\tau_{i+1} - \tau_i) \tag{3.4}$$

is valid for all $u(t) \in U$, $v(t) \in V$. Here $O(\alpha)$ is an infinitely small quantity of order not lower than that of α).

The same estimates enable us to verify the validity of the following lemma.

Lemma 3.2. Let $M \subset W(t)$, $W(\vartheta) = M$ and $x[t_0] = x_0 \in W(t_0)$. If the sets $W(t)$ $(t_0 \leqslant t \leqslant \vartheta)$ are stable, the extremal strategy ensures that

$$\varepsilon_0(\mathbf{U}^{(e)}) = 0 \tag{3.5}$$

In other words, if the sets $W(t)$ are stable for all sufficiently small values of Δ, then the controls $u_\Delta^{(e)}[t]$ chosen by the extremal strategy $\mathbf{U}^{(e)}$ ensure that the phase point $x[t]$ behaves in such a way that it occurs at least once in an arbitrarily small prespecified neighborhood of the set M for $t \leqslant \vartheta$ whatever the permissible behavior of the second player.

Note 3.1. The strategy $\mathbf{U}^{(e)}$ constructed in Sect. 2 is an approximating strategy which generates a piecewise-constant control $u_\Delta^{(e)}[t]$ of the form (1.3). However, under certain conditions this strategy can be formalized within the framework of differential equations in contingencies, as is done in [10–12, 18]. Specifically, let the vector s occurring in condition (2.4) be unique for each state x not occurring in the set $W(t)$. We use the symbol $F^{(1)}(t, x)$ to denote the convex shell of the set which the vector $f^{(1)}(t, x, u)$ as the vector u runs through U. The symbol $F^{(e)}(t, x)$ denotes the set of all vectors $f = f^{(e)}$ from $F^{(1)}$ satisfying the condition

$$s' f^{(e)} = \max s' f \qquad (f \in F^{(1)}) \tag{3.6}$$

if x is not contained in $W(t)$, and only the condition

$$f^{(e)} \in F^{(1)} \tag{3.7}$$

if $x \in W(t)$.

We define the motion $x[t]$ $(t_0 \leqslant t \leqslant \vartheta)$ of system (1.1) for some control $v = v[t]$ and for a generalized control u dictated by the strategy $\mathbf{U}^{(e)}$ defined by the sets $F^{(e)}(t, x)$ as any absolutely continuous function $x[t]$ which satisfies the condition

$$\frac{dx[t]}{dt} \in F^{(e)}(t, x[t]) + f^{(2)}(t, x[t], v[t]) \tag{3.8}$$

for almost all values of t.

The right side of (3.8) contains a set of vectors f of the form $f = f^{(e)} + f^{(2)}$ for $f^{(e)} \in F^{(e)}$. By virtue of the above condition of uniqueness of the vector s which is fulfilled, for example, if the sets $W(t)$ are convex, the extremal strategy $\mathbf{U}^{(e)}$ in the case of strongly stable sets $W(t)$ ensures that

$$\rho(x[t], W(t)) = 0 \quad (t_0 \leqslant t \leqslant \vartheta) \tag{3.9}$$

provided $x[t_0] \in W(t_0)$ whatever the behavior of the second player. Fulfillment of condition (3.9) is also ensured by the extremal strategy $\mathbf{U}^{(e)} \div F^{(e)}(t, x)$ in those cases where the second player is also using some generalized strategy $\mathbf{V}$ described by the sets $F^{(\mathbf{V})}(t, x)$, so that the motion $x[t]$ is defined by the contingency

$$\frac{dx[t]}{dt} \in F^{(e)}(t, x[t]) + F^{(\mathbf{V})}(t, x[t]) \tag{3.10}$$

(The symbol $\div$ represents the correspondence between the strategy and the sets defining it).

A similar remark can be made under the conditions of Lemma 3.2 by imposing the additional condition of uniqueness of the vector s in (1.3).

4. Lemmas 3.1 and 3.2 imply that a strategy $\mathbf{U}$ which would guarantee convergence of the point $x[t]$ to the set M can be constructed on the basis of the extremal strategy $\mathbf{U}^{(e)}$ provided one can find the sets $W(t)$ $(t_0 \leqslant t \leqslant \vartheta)$ satisfying the conditions of these lemmas (and also the condition $W(\vartheta) = M$ in the case of Lemma (3.1). Such sets are derivable from constructions such as those described in [5−7], although the effective use of these constructions is somewhat difficult. We can approach the construction of the required sets $W(t)$ in a somewhat different way by using the notion of absorption of the set M by process (1.1) at the instant ϑ or by the instant ϑ (from a given position $\{t_*, x_*\}$, $t_0 \leqslant t_* \leqslant \vartheta$).

This approach to game problems was proposed in [13] in connection with the consequent extremal aiming rule described in Sect. 2 in the modified form of the extremal strategy $U^{(e)}$; it was later developed in [10−14]. In this case, however, the properties of strong stability or stability of the sets $W(t)$ $(t_0 \leqslant t \leqslant \vartheta)$ in the general case no longer follow automatically from the method of constructing these sets. One or the other of these properties must be additionally verified on the basis of the specific properties of the system (1.1) and the set M.

We begin by citing two definitions which serve as a basis for describing the sets $W(t)$ and then writing out sufficient conditions whose fulfillment ensures that these sets $W(t)$ have the necessary stability properties.

Definition 4.1. We say that the set M is absorbed per program by process (1.1),(1.2) from the position $\{t_*, x_*\}$ at an instant $\vartheta \geqslant t_*$ if for any permissible function $v(t)$ $(t_* \leqslant t < \vartheta)$ we can choose a permissible function $u(t)$ $(t_* \leqslant t < \vartheta)$ such that the motion $x(t)$ $(x(t_*) = x_*)$ generated by the chosen pair of controls $\{u(t), v(t)\}$ satisfies the condition

$$x(\vartheta) \in M \tag{4.1}$$

Definition 4.2. We say that the set M is absorbed per program by process (1.1), (1.2) from the position $\{t_*, x_*\}$ by an instant $\vartheta \geqslant t_*$ if for any permissible function $v(t)$ $(t_* \leqslant t < \vartheta)$ we can choose a permissible function $u(t)$ $(t_* \leqslant t < \vartheta)$ such that the motion $x(t)$ $(x(t_*) = x_*)$ generated by the chosen pair of controls $\{u(t), v(t)\}$ satisfies the condition

$$\min_t \rho\ (x,(t),\ M) = 0 \qquad (t_* \leqslant t \leqslant \vartheta) \tag{4.2}$$

Let us assume that the number t_0 is fixed and that the number $\vartheta \geqslant t_0$ has been chosen in some way. We use the symbol $W(t, \vartheta)$ $(t_0 \leqslant t \leqslant \vartheta)$ to denote the set of all points x for which the set M is absorbed per program by process (1.1), (1.2) from the position $\{t, x\}$ at the instant ϑ. If some sets $W(t)$ $(t_0 \leqslant t \leqslant \vartheta)$ are strongly stable and if $M = W(\vartheta)$, then

$$W(t) \subset W(t, \vartheta) \qquad (t_0 \leqslant t \leqslant \vartheta) \tag{4.3}$$

However, the sets $W(t, \vartheta)$ need not be strongly stable.

Next we use the symbol $W^*(t, \vartheta)$ $(t_0 \leqslant t \leqslant \vartheta)$ to denote the set of all points x for which the set M is absorbed per program by process (1.1), (1.2) from the position $\{t, x\}$ by the instant ϑ. If some sets $W(t)$ $(t_0 \leqslant t \leqslant \vartheta)$ are stable and if $W(\vartheta) = M$, then

$$W(t) \subset W^*(t, \vartheta) \qquad (t_0 \leqslant i \leqslant \vartheta) \tag{4.4}$$

However, the sets $W^*(t, \vartheta)$ need not be stable.

By virtue of Lemmas 3.1 and 3.2 and conditions (4.3), (4.4) the question of the conditions under which the sets $W(t, \vartheta)$ or the sets $W^*(t, \vartheta)$ are strongly stable or stable, respectively, has an important bearing on the problem of convergence of the point $x[t]$ to the set M. In fact, the strong stability or stability of the respective sets implies (by virtue of Lemmas 3.1 and 3.2) that it is sufficient to take the extremal strategy $U^{(e)}$ based on the sets $W(t, \vartheta)$ or $W^*(t, \vartheta)$, respectively, as the required strategy U which ensures convergence (1.4) or (1.5) of the point $x[t]$ to the set M. The following two lemmas specify certain sufficient conditions of strong stability of the sets $W(t, \vartheta)$ or stability of the sets $W^*(t, \vartheta)$.

We begin by setting out certain assumptions.

Condition 4.3. Assuming that the set U is convex, we stipulate (see [15]) that the vector $f^{(1)}(t,x,u)$ runs through the convex set $F^{(1)}(t,x)$ as the vector u runs through U for all $\{t, x\}$.

Let us suppose that we have chosen some position $\{t_*, x_*\}$, some number $\Delta>0$, and some permissible function $v(t)$ $(t_* \leqslant t < t_* + \Delta)$. Let us consider the motions $x(t)$ $(t_* \leqslant t \leqslant t_* + \Delta,\ x(t_*) = x_*)$ generated by the given control $v = v(t)$ and by all the possible permissible controls $u = u(t)$ $(t_* \leqslant t < t_* + \Delta)$. We denote the resulting set of points $x = x(t_* + \Delta)$ by the symbol $X(t_*, x_*, \Delta, v(t))$.

Condition 4.4. The set $X(t_*, x_*, \Delta, v(t))$ is convex for all $\{t_*, x_*\}$, $v(t)$ and $\Delta \leqslant \Delta_0$, where Δ_0 is a sufficiently small positive number.

Let $x_* \in W(t_*, \vartheta)$. Then for any permissible function $v(t)$ $(t_* \leqslant t < \vartheta)$ there exists a permissible function $u(t)$ $(t_* \leqslant t < \vartheta)$ which together with $v(t)$ brings the motion $x(t)$ $(x(t_*) = x_*)$ to the point $x(\vartheta) \in M$. On the other hand, if $x_* \in W^*(t, \vartheta)$, then for any permissible function $v(t)$ $(t_* \leqslant t < \vartheta)$ there exists a permissible function $u(t)$ $(t_* \leqslant t < \vartheta)$ which together with $v(t)$ generates a motion $x(t)$ satisfying the condition $x(t^*) \in M$ $(t^* \leqslant \vartheta)$. We assume that this procedure enables us to establish a certain correspondence between the permissible functions $v(t)$ $(t_* \leqslant t < \vartheta)$ and the motions $x(t)$ $(t_* \leqslant t \leqslant \vartheta)$. We denote this correspondence by the symbol $v(t) \to x(t)$.

Now let us suppose that a certain point x^* does not belong to $W(t_* + \Delta, \vartheta)$. Then, there exists a permissible function $v^*(t)$ $(t_* + \Delta \leqslant t < \vartheta)$ such that the motion $x(t)$ $(x(t_* + \Delta) = x^*)$ generated by the controls $\{u(t), v^*(t)\}$ does not arrive at M at the instant ϑ for any chosen permissible function $u(t)$ $(t_* + \Delta \leqslant t < \vartheta)$. If the point x^* does not occur in $W^*(t_* + \Delta, \vartheta)$, then there exists a permissible function $v^*(t)$ $(t_* + \Delta \leqslant t < \vartheta)$ such that the motion $x(t)$ generated by the controls $\{u(t), v^*(t)\}$ $(x(t_* + \Delta) = x^*)$ does not arrive at M for any t from the segment $[t_* + \Delta, \vartheta]$ no matter what permissible function $u(t)$ $(t_* + \Delta \leqslant t < \vartheta)$ is chosen.

Let us suppose that this enables us to establish a certain correspondence between the points x^* and the functions $v^*(t)$ $(t_* + \Delta \leqslant t < \vartheta)$. We denote this correspondence by the symbol $x^* \to v^*(t)$. Further, let the function $v(t)$ be defined in some way for $t_* \leqslant t < t_* + \Delta$, and let it coincide with some function $v^*(t)$ for $t_* + \Delta \leqslant t < \vartheta$.

Then, by virtue of what we have already said, the chosen position $\{t_*, x_*\}$ can be matched with a mapping $x^* \to x(t_* + \Delta)$ which is defined by the two corresondences $x^* \to v^*(t)$ and $v(t) \to x(t)$ $(v(t) = v^*(t)$ for $t_* + \Delta \leqslant t < \vartheta)$.

Condition 4.5. Whatever the position $\{t_*, x_*\}$ $(x_* \in W(t_*, \vartheta)$ or $x_* \in W_*(t_*, \vartheta))$, the function $v(t)$ $(t_* \leqslant t < t_* + \Delta)$, and the sufficiently small number $\Delta > 0$, it is possible to choose a mapping $x^* \to x(t_* + \Delta)$ which is continuous (i.e. in domains where x^* does not occur in $W(t_* + \Delta, \vartheta)$ or in $W^*(t_* + \Delta, \vartheta)$, respectively).

The following statements are valid.

Lemma 4.1. If Conditions 4.3–4.5 are satisfied, the sets $W(t, \vartheta)$ are strongly stable.

In fact, let us suppose that this is not the case. Then for some position $\{t_*, x_*\}$, where $x_* \in W(t_*, \vartheta)$, for some arbitrarily small $\Delta > 0$, and for a suitably chosen $v(t)$ $(t_* \leqslant t < t_* + \Delta)$ the set $X(t_*, x_*, \Delta, v(t))$ does not have points in common with $W(t_* + \Delta, \vartheta)$. (We must bear in mind that by Condition 4.3 the set $X(t_*, x_*, \Delta, v(t))$, as well as all the sets $W(t, \vartheta)$ are closed (see [15, 18])). But Condition 4.5 can then be used to construct a continuous mapping of this set into itself. According to a familiar theorem (see [16], p. 296, Theorem 5) this mapping has a fixed point x°. But the preceding constructions imply that the motion $x(t)$ passes through this point for $t = t_* + \Delta$; on the one hand this motion arrives at the set M at the instant ϑ; on the other hand this motion $x(t)$ cannot arrive at this set M at the instant ϑ. The resulting contradiction proves the lemma.

The following lemma can be proved in similar fashion.

Lemma 4.2. If Conditions 4.3 – 4.5 are fulfilled, the sets $W^*(t, \vartheta)$ are stable.

The following statement follows directly from Lemmas 3.1, 3.2, 4.1 and 4.2.

Theorem 4.1. Let $x_0 \in W(t_0, \vartheta)$ or $x_0 \in W^*(t_0, \vartheta)$. If Conditions 4.3–4.5 are satisfied, the extremal strategy $U^{(e)}$ based on the sets $W(t, \vartheta)$ or $W^*(t, \vartheta)$ guarantees convergence of the point $x[t]$ with the set M either at the instant ϑ or by the instant ϑ, respectively.

Note 4.1. Condition 4.5 can be generalized somewhat in the followng way. First, the correspondence $v(t) \to x(t)$ can be replaced by the correspondence $v(t) \to \{x(t)\}$, where $\{x(t)\}$ is no longer the single motion $x(t)$, but rather a whole family of motions $\{x(t)\}$, each of which has the necessary property of convergence with the set M. The correspondence $x^* \to v^*(t)$ can likewise be replaced by the correspondence $x^* \to \{v^*(t)\}$, where $\{v^*(t)\}$ is again some set of controls $v^*(t)$, each of which ensures the required deviation of the motions $x(t)$ from M. As above, these two correspondences define the mapping $x^* \to, \to \{x(t_* + \Delta)\}$ of the points x^* onto the sets $\{x(t_* + \Delta)\}$. We can now replace our original Condition 4.5 by the following statement.

Condition 4.5*. Whatever the position $\{t_*, x_*\}$ ($x_* \in W(t_*, \vartheta)$ or $x_* \in W^*(t_*, \vartheta)$), the function $v(t)$ $(t_* \leqslant t \leqslant t_* + \Delta)$, and the sufficiently small number $\Delta > 0$, the mapping $x^* \to \{x(t_* + \Delta)\}$ can be chosen in such a way that the sets $\{x(t_* + \Delta)\}$ are convex, closed, and semicontinuous above the inclusion (by the variation of x^* in domains where x does not occur either in $W(t_* + \Delta, \vartheta)$ or in $W^*(t_* + \Delta, \vartheta)$, respectively).

Lemmas 3.1, 3.2, and Theorem 4.1 remain valid upon replacement of Condition 4.5 by Condition 4.5*. However, Lemmas 4.1 and 4.2 must now be proved not on the basis of Theorem 5 of [16], but rather by means of the theorem of [17] whereby Condition 4.5* implies the existence of a point $x^\circ \in \{x(t_* + \Delta)\}^\circ$, where $x^\circ \to \{x(t_* + \Delta)\}^\circ$, for the mapping $x^* \to \{x_*(t_* + \Delta)\}$, and this again yields the necessary contradiction.

5. The formulation of Condition 4.5, which plays a fundamental role in the hypotheses of Theorem 4.1, is too general for ready verification. In this section we shall cite a certain condition which throws some light on the circumstances of Condition 4.5. To be specific, we shall confine our attention to the case of absorption of the set M at the instant ϑ. The case of absorption of the set M by the instant ϑ entails similar analysis of system (5.1) below, but requires allowance for certain additioanl details.

Let us suppose that Eq. (1.1) is of the form

$$dx/dt = f(t, x) + B(t)u + C(t)v \tag{5.1}$$

where $B(t)$ and $C(t)$ are continuous matrices of functions of the corresponding dimensions. We assume that the sets U and V in conditions (1.2) are convex. Condition 4.3 is then satisfied automatically. By transforming system (5.1) we can always ensure that the set U contains a zero vector. This allows us to assume that the indicated condition is fulfilled.

Let us suppose that the point x^* does not occur in the set $W(t^*, \vartheta)$. This enables us to find the pair of controls $\{u^*(t), v^*(t)\}$ which solve the problem

$$\rho(x^*(\vartheta), M) = \max_{v(t)} \min_{u(t)} \rho(x(\vartheta), M) \tag{5.2}$$

where $x^*(t)$ $(t^* \leqslant t \leqslant \vartheta,\ x(t^*) = x^*)$ is the motion generated by the controls $\{u^*(t), v^*(t)\}$; the maximin must be computed over all the integrable functions $u(t) \in U$, $v(t) \in V$ $(t^* \leqslant t < \vartheta)$. We infer from the definition of the set $W(t^*, \vartheta)$ and from the choice of the point x^* outside this set that the resulting maximin yields a positive value of $\rho(x^*(\vartheta), M)$.

Condition 5.6. The function $v^*(t)$ which yields the solution of problem (5.2) is unique (essentially over the half-interval $[t^*, \vartheta)$) for any position $\{t^*, x^*\}$, where x^* does not occur in $W(t^*, \vartheta)$.

Now let the point $x_* \in W(t_*, \vartheta)$ and let us suppose that we have chosen some permissible function $v(t)$ $(t_* \leqslant t < \vartheta)$. Let us find the smallest of the numbers μ for which there exists a function $u_\mu(t) \in \mu U$ $(t_* \leqslant t < \vartheta)$ satisfying the condition

$$\rho(x_\mu(\vartheta), M) = 0 \tag{5.3}$$

Here $x_\mu(t)$ $(t_* \leqslant t \leqslant \vartheta,\ x_\mu(t_*) = x_*)$ is the motion generated by the controls $\{u_\mu(t), v(t)\}$; the symbol μU represents the set of vectors of the form $u = \mu u^*$, where $u^* \in U$. From the definition of the set $W(t_*, \vartheta)$ and the choice of the point x_* from this set we infer that μ°, the smallest of the numbers μ for which Eq. (5.3) can be fulfilled, is not larger than unity.

Condition 5.7. The solution of problem (5.3) for $\mu = \mu^\circ$ is attained on the unique motion $x_{\mu^\circ}(t)$ for any position $\{t_*, x_*\}$, where $x_* \in W(t_*, \vartheta)$ and for any permissible function $v(t)$.

The following statement is valid.

Lemma 5.1. Fulfillment of Conditions 5.6 and 5.7 implies fulfillment of Condition 4.5.

In fact, let us set $t^* = t_* + \Delta$. The relation $x^* \to v^*(t)$ under Condition 5.6 can be determined by choosing precisely the solution of problem (5.2) as our $v^*(t)$. The relation $v(t) \to x(t)$ can be determined from Condition 5.7 by taking the motion $x_{\mu^\circ}(t)$ as our $x(t)$.

We must now verify whether the mapping $x^* \to x_{\mu^\circ}(t_* + \Delta)$ is continuous. Let us suppose that this is not the case. This implies the existence of a sequence of points $x^{(i)}(i = 1, 2, \ldots)$ which converge to the point x^* and are such that the corresponding points $x_{\mu^\circ}^{(i)}(t_* + \Delta)$ do not approach the point $x_{\mu^\circ}(t_* + \Delta)$. We can select a weakly convergent subsequence from the sequence of functions $v^{(i)}(t)$ $(t_* + \Delta \leqslant t < \vartheta)$.

We can show that the weak limit $v^\infty(t) \in V$ $(t_* + \Delta \leqslant t < \vartheta)$ of this subsequence is the solution of problem (5.2) for the point x^*. Then, by Condition 5.6, $v^\infty(t) = v^*(t)$. This enables us to isolate from the sequence $x_{\mu^\circ}^{(i)}(t)$ a subsequence which converges to the function $x_{\mu^\circ}(t)$, which by Condition 5.7 is the unique function which solves problem (5.3) for $v(t) = v^\infty(t)$ $(t_* + \Delta \leqslant t < \vartheta)$. But this contradicts our supposition. The contradiction proves the lemma.

Note 5.1. It is useful to compare Conditions 5.6 and 5.7 with those sufficient conditions of strong stability of the sets $W(t, \vartheta)$ which are known to be valid for linear systems provided the set M is convex (e.g. see [10–12, 18]). These conditions can be formulated as follows.

Let system (1.1) be described by the linear equation

$$dx/dt = A(t)x + B(t)u - C(t)v + f(t) \qquad (5.4)$$

Moreover, let the sets U and V in conditions (1.2) be convex and let the set M also be bounded and convex. The point x then occurs in $W(t_*, \vartheta)$ if and only if

$$\varkappa(t_*, \vartheta, l) + l' \Phi(\vartheta, t_*) x \geqslant 0 \qquad (5.5)$$

whatever the unit vector l. Here the function $\varkappa(t_*, \vartheta, l)$ is defined by the equation

$$\varkappa(t_*, \vartheta, l) = \varkappa^{(1)}(t_*, \vartheta, l) - \varkappa^{(2)}(t_*, \vartheta, l) + l' \int_{t_*}^{\vartheta} \Phi(\vartheta, \tau) f(\tau)\, d\tau \qquad (5.6)$$

where $\varkappa^{(2)}(t_*, \vartheta, l)$ and $\varkappa^{(1)}(t_*, \vartheta, l)$ are the support functions of the attainability domains from the position $\{t_*, x(t_*) = 0\}$ for the motion $x(t)$ (5.4) by way of $v(t) \in V$ and $u_\delta(t) = u(t) + p\delta(t - \vartheta)$, where $u(t) \in U$u $- p \in M$. The symbol $\delta(t)$ represents the δ-function

$$\varkappa^{(1)}(t_*, \vartheta, l) = \max_{u(\tau) \in U} \left(l' \int_{t_*}^{\vartheta} \Phi(\vartheta, \tau) B(\tau) u(\tau)\, d\tau \right) + \max_{-p \in M} l'p \qquad (5.7)$$

$$\varkappa^{(2)}(t_*, \vartheta, l) = \max_{v(\tau) \in V} \left(l' \int_{t_*}^{\vartheta} \Phi(\vartheta, \tau) C(\tau) v(\tau)\, d\tau \right) \qquad (5.8)$$

where $\Phi(t, \tau)$ is the fundamental matrix of solutions for the homogeneous equation $dx/dt = A(t)x$. The results of [10–12] imply that the sets $W(t, \vartheta)$ are strongly stable if, provided that

$$\min_{\|l\|=1} (\varkappa(t, \vartheta, l) + l'\Phi(\vartheta, t) x + \varepsilon) = 0 \qquad (5.9)$$

where $\varepsilon > 0$, the minimum in the left side of (5.9) is attained on the unique vector l (the symbol $\| l \|$ represents the Euclidean norm of the vector l).

Fulfillment of Conditions 5.6 and 5.7 clearly implies the fulfillment of this condition. Generally speaking, however, this condition is somewhat weaker than Conditions 5.6 and 5.7.

On the other hand, making use of the more general theorem concerning a fixed point in the case of multivalued mappings [17] as we did in Note 4.1, we obtain strong stability conditions which are entirely analogous to the above conditions of uniqueness of the minimizing vector l in (5.9).

We also note that maximum condition (2.4) differs from the extremal rule [10–12] in minor details only.

Finally, we emphasize that the sets $W(t, \vartheta)$ in the above linear case are necessarily convex. The vector s in condition (2.4) is therefore necessarily unique, so that (in accordance with Note 3.1) the extremal strategy $U^{(e)}$ is also formalizable within the framework of differential equations in contingencies in all cases.

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